The solution of the Falkner-Skan equation arising in the modelling of boundary-layer problems via variational iteration method

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January 19, 2010

Abstract: In this work the well-known Falkner-Skan equation is considered. This equation appears in the modelling of various phenomena in physics and engineering. The He's variational iteration method which is a very efficient tool for solving different kinds of problems, is employed for solving this problem. Some other approaches are introduced to compare the efficiency of the new procedure. Several test examples are given to show the advantages of the present method over other existing techniques.

Keywords: He's variational iteration method, Falkner-Skan boundary layer equation, Padé approximation.

1 Introduction

The Variational Iteration Method (VIM) was proposed by Ji-Huan He in 1991. This method is an effective and flexible procedure for solving linear and nonlinear problems and has been applied for solving various kinds of problems in science and engineering. For instance, linear and nonlinear system of ordinary differential equations, Blasius equation [46], problems in calculus of variations [50], Helmholtz equation [31] and Thomas-Fermi equation [11] are solved by this method. For more information about variational iteration method see [27, 32, 33, 34, 13, 41, 42, 44, 45, 14, 47, 48, 54, 55, 16, 15, 18, 19, 17, 51]. Also authors of [49] investigated the convergence of

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the VIM. With the best of our knowledge [49] is the only published research work about the convergence of VIM.

Now consider the general differential equation:

$$Lu + Nu = g(x,t),$$

where L and N are linear and nonlinear operators respectively and $g(x,t)$ is the inhomogeneous term of equation. According to the VIM, we construct an iteration formulation in the following way:

$$u_{n+1} = u_n + \int_0^x \lambda(Lu_n(s) + N\tilde{u}_n(s) - g(s,t))ds, \tag{1.1}$$

where λ is Lagrange multiplier which should be identified. The function $\tilde{u}_n$ is restricted variation such that $\delta\tilde{u}_n = 0$. The subscript n denotes the nth approximation, in fact (1.1) is a correction functional. The Lagrange multiplier λ is found in a way that the variation of the right hand side in (1.1) be zero.

In 1931 Falkner and Skan [23] developed similarity transformation method for the two-dimensional wedge flows on a two-dimensional incompressible laminar boundary layer equation. They introduced a one-dimensional, third order, nonlinear boundary value problem named "Falkner-Skan equation" given by

$$\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} + \beta[1 - (\frac{df}{d\eta})^2] = 0, \qquad 0<\eta<\infty, \tag{1.2}$$

subject to the boundary conditions

$$f(0) = \delta, \qquad \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}_{\eta\to\infty} = 1, \tag{1.3}$$

where parameters $\beta > 0$ and δ are known. This ordinary differential equation appears in the modelling of boundary layer problems for the two-dimensional steady and incompressible laminar flows passing a wedge in a common area of interest. Uniqueness and detailed analysis of solution to the Falkner-Skan equation are investigated in [35, 52]. We state the following result about existence and uniqueness of this problem which can be derived from some well-known results in [25];

Theorem 1.1. *(a) Eq. (1.2) with (1.3) and the side condition* $0 < f'(\eta) < 1$ *has a unique solution for* $\beta \geq 0$. *Moreover, the solution satisfies in*

$$f''(\eta) > 0, \qquad for \quad \eta \in (0,\infty). \tag{1.4}$$

(b) There exist $\beta^* \in (-\infty, 0)$ *such that (1.2) and (1.3) with* $0 < f'(\eta) < 1$ *has at least one solution for* $\beta \in (\beta^*, 0)$ *and the solutions are not unique and satisfy (1.4).*
(c) Eq. (1.2) subject to (1.3) with $0 < f'(\eta) < 1$ *has a unique solution for* $\beta = \beta^*$ *and the solution satisfies (1.4).*
(d) Eq. (1.2) with (1.3) and $0 < f'(\eta) < 1$ *has no solution for* $\beta < \beta^*$*.*

The interested reader can see [35] for some new results on existence and uniqueness of the solutions of the Falker–Skan equation.

This type of boundary layer problems can't be solved directly in a closed form generally. Therefore, the researchers emphasize on the numerical methods to solve it. The first numerical method for the problem was presented by [12, 24]. In [3, 4] some finite difference methods were used for solving this problem.

Shooting methods also were implemented by authors of [7, 30]. As some other investigations we can mention the differential transformation method [28], Homotopy perturbation method [53], finite element method [5], pseudo-spectral method [21, 36], Adomian decomposition method [1, 22]. For more information about this equation, the interested reader can see [9, 29]. Authors of [39] used analysis of asymptotic behavior of the solution at ∞ to solve this equation. Salama [38] proposed a one-step technique of the order five to solve this equation. Recently author of [6] developed the automatic differentiation to find the solution of the Falker–Skan equation. Using this method which is neither numerical or symbolic, a Taylor series solution is constructed for the initial value problems by calculating the Taylor coefficients recursively.

This paper uses another approach. The direction of this paper is the study of Falkner–Skan equation via He's variational iteration method. The rest of the current paper is arranged as follows:

Section 2 contains some approaches suggested to transform the given equation to some other forms to make it easier to handle the model. In Section 3 we aim to apply the variational iteration method (VIM) in a direct manner to establish $f''(0)$ for Falkner–Skan equation, then some modification of it are given to refine the solution of Falkner–Skan equation. In Section 4, some illustrative examples are given. Section 5 ends this report with some concluding remarks.

2 The Falkner-Skan boundary layer equation

Nature of Falkner–Skan boundary layer equation refers to the flow of an incompressible viscous fluid over a wedge. In fact this equation describes the class of similar laminar flows in boundary layer on a permeable or impermeable wall, for example flows along curvilinear profiles such as airplane wings.

Assume u and v are respective velocity components in the x and y directions of the fluid flow, ν is the kinematic viscosity of the fluid and w is the reference velocity at the edge of the boundary layer that depends on x.

The continuity and momentum equations that refer to conservation of mass and conservation of momentum laws of movement of the fluid over wedge, are respectively:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = w\frac{dw}{dx} + \nu\frac{\partial^2 u}{\partial y^2},$$

with boundary conditions:

$$at \;\; y = 0: \qquad u = v = 0, \qquad\qquad as \;\; y \to \infty: \qquad u \to w(x) = w_\infty(\frac{x}{L})^m,$$

where w_∞ is the mean stream velocity, L is the length of the wedge and m is the Falkner–Skan power-low parameter. Now the stream function $\psi(x, y)$ can be introduced such that:

$$u = \frac{\partial \psi}{\partial y}, \qquad and \qquad v = -\frac{\partial \psi}{\partial x}. \tag{2.1}$$

By substituting this definition in the momentum equation, then integrating equations (2.1) and introducing a similarity variable and similarity function by:

$$\eta = \sqrt{\frac{1+m}{2}\frac{w_\infty}{\nu L^m}}\frac{y}{x^{\frac{1-m}{2}}}, \qquad f(\eta) = \sqrt{\frac{1+m}{2}\frac{L^m}{\nu w_\infty}}\frac{\psi}{x^{\frac{1+m}{2}}},$$

and substituting them into the transformed momentum equation, we get the well-known Falkner–Skan equation

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta[1 - (\frac{df}{d\eta})^2] = 0, \qquad 0<\eta<\infty, \tag{2.2}$$

subject to the boundary conditions

$$f(0) = 0, \qquad \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}_{\eta\to\infty} = 1, \tag{2.3}$$

where $\beta = \frac{2m}{m+1}$ [28, 43]. But we can assume $f(0) = \delta$ where the mass-transfer parameter δ sets the measure for the mass flow rate through the wall boundary direction, positive value determines flow with suction and negative with injection through the wall boundary. The zero value corresponds to flow along impermeable wall with zero mass transfer.

It is customary to replace the third equation of (2.3) with the condition:

$$\frac{df}{d\eta}(\eta_\infty) = 1,$$

for some sufficiently large values of η_∞ which are determined experimentally.

Notice that this equation is defined in domain $[0, \eta_\infty]$. Now the following transform is made:

$$\xi = \frac{\eta}{\eta_\infty}, \qquad g = \frac{f}{\eta_\infty}.$$

This leads to the following equations:

$$\frac{df}{d\eta} = \frac{dg}{d\xi}, \qquad \eta_\infty \frac{d^2 f}{d\eta^2} = \frac{d^2 g}{d\xi^2}, \qquad {\eta_\infty}^2 \frac{d^3 f}{d\eta^3} = \frac{d^3 g}{d\xi^3}.$$

And consequently we have

$$\frac{d^3 g}{d\xi^3} + \eta_\infty^2 g \frac{d^2 g}{d\xi^2} + \eta_\infty^2 \beta [1 - (\frac{dg}{d\xi})^2] = 0, \qquad 0<\xi<1, \tag{2.4}$$

subject to the boundary conditions

$$g(0) = \frac{\delta}{\eta_\infty}, \qquad \frac{dg}{d\xi}(0) = 0, \qquad \frac{dg}{d\xi}(1) = 1. \tag{2.5}$$

Also we refer the interested reader to [1, 9, 3, 4, 7, 5, 6, 12, 21, 25, 24, 30, 28, 35, 43, 36, 52, 53, 46] for more information on this equation and its derivation.

3 The variational iteration method

The VIM will be implemented for the transformed Falkner–Skan equation (2.4). Consider the following correction functional:

$$g_{n+1}(\xi) = g_n(\xi) + \int_0^\xi \lambda(g_n'''(s) + \eta_\infty^2 \tilde{g}_n(s)\tilde{g}_n''(s) + \eta_\infty^2 \beta(1 - \tilde{g}_n'^2(s)))ds. \tag{3.1}$$

According to [27] we have $\lambda = -\frac{1}{2}(s-\xi)^2$ and therefore, we have

$$g_{n+1}(\xi) = g_n(\xi) - \int_0^\xi \frac{1}{2}(s-\xi)^2 (g_n'''(s) + \eta_\infty^2 g_n(s) g_n''(s) + \eta_\infty^2 \beta (1 - g_n'^2(s)))ds, \tag{3.2}$$

with a suitable choice of g_0. Let us point out the main hurdle in the solution of the Falkner–Skan equation is the absence of the second derivative $g''(0)$. In fact all of the higher derivatives of $g(\xi)$ (solution of equation) at $\xi = 0$ will be determined by $g''(0)$ using Taylor expansion [37].

The initial term $g_0(\xi)$ usually is considered as a polynomial. For the present problem we consider the initial term as

$$g_0(\xi) = g(0) + \xi g'(0) + \frac{\xi^2}{2!} g''(0) - \frac{\xi^3}{3!} \beta \eta_\infty^2,$$

or equivalently

$$g_0(\xi) = \frac{\delta}{\eta_\infty} + \frac{\xi^2}{2!}\alpha - \frac{\xi^3}{3!}\beta\eta_\infty^2,$$

where $\alpha = g''(0)$ is unknown. For a sufficiently large value of n, $g_n(\xi)$ is an accurate approximation of $g(\xi)$. The unknown α is found in a way that

$$g_n'(1) = 1,$$

which results a nonlinear equation.

Unfortunately as we will see in next section, VIM becomes weaker by increasing η_∞. So aside from VIM some modifications of it can be implemented to overcome this demerit. Now two modifications of VIM are introduced:

A) For large η_∞ such as $\eta_\infty = 6$ the results for α via VIM are inaccurate. Then we apply Padé approximation on the obtained g_n as Wazwaz did in [47]. Moreover the diagonal approximant is the most accurate approximant. Therefore we will construct only the diagonal approximations [M/M]. This causes that convergence region increases. We assume that [L/M] denotes the Padé approximation to $g_n(t)$ as is defined in Eq. (15) of the paper [10].

B) Errors of solution for large η_∞ is very high. Now a suitable refinement for VIM is used. In this way at first $\alpha = g''(0)$ is computed by VIM. By the substitution of α in g_0, the initial guess of MVIM is constructed. This causes that the maximum value of the residual of Falkner–Skan equation decreases, although $\alpha = g''(0)$ is the same as before.

4 Numerical results and discussion

In this section the VIM is implemented, as was described in the previous section, for values of β and η_∞ and $g(0)$ that were reported in [2]. These values are showed in Examples 1-9:

Example 1.
In this example the Falkner–Skan equation is considered for $g(0) = 0$ and $\beta = 0.5$ and $\eta_\infty = 2$ □
Example 2.
In this example the Falkner–Skan equation is considered for $g(0) = 0$ and $\beta = 0.5$ and $\eta_\infty = 4$ □
Example 3.
In this example the Falkner–Skan equation is considered for $g(0) = 0$ and $\beta = 0.5$ and $\eta_\infty = 6$ □
Example 4.
In this example the Falkner–Skan equation is considered for $g(0) = 0.05$ and $\beta = 0.25$ and $\eta_\infty = 2$ □
Example 5.
In this example the Falkner–Skan equation is considered for $g(0) = 0.02$ and $\beta = 0.5$ and $\eta_\infty = 4$ □
Example 6.
In this example the Falkner–Skan equation is considered for $g(0) = 0.01$ and $\beta = 0.5$ and $\eta_\infty = 2$ □
Example 7.
In this example the Falkner–Skan equation is considered for $g(0) = -0.5$ and $\beta = 0.25$ and $\eta_\infty = 2$ □
Example 8.
In this example the Falkner–Skan equation is considered for $g(0) = -0.25$ and $\beta = 0.5$ and $\eta_\infty = 4$ □
Example 9.
In this example the Falkner–Skan equation is considered for $g(0) = -0.15$ and $\beta = 0.25$ and $\eta_\infty = 4$ □

Results of $\alpha = g''(0)$ for these values are computed and are shown in Table 1. As we see for large η_∞ such as $\eta_\infty = 6$ the results of VIM are very different from the results of [2]. So the Padé approximation is used to overcome this difficulty. Table 2 shows the value of α for $\eta_\infty = 6$. The VIM with Pade approximation can be more successful than the classic VIM.

Table 1: Results of the variational iteration method versus the method of [2].

$g(0)$	η_∞	β	$\alpha[VIM]$	$\alpha[2]$
0	2	0.5	1.910511622	1.910152880
0	4	0.5	3.478838064	3.452202391
0	6	0.5	6.687805351	5.469609324
0.05	2	0.25	1.695718722	1.698115783
0.02	4	0.5	3.376351676	3.431547475
0.01	2	0.5	1.933744867	1.597446553
-0.5	2	0.25	0.701401995	0.708267207
-0.25	4	0.5	1.960524671	1.965660199
-0.15	4	0.25	1.570261336	1.522188630

Table 2: The results of the variational iteration method using Padé approximation method versus the method of [2] for $\eta_\infty = 6$.

$g(0)$	η_∞	β	α [Padé approximation]	$\alpha[2]$
0	6	0.5	5.615188503	5.469609234

Moreover we define:

$$R_g(\xi) = \frac{d^3g}{d\xi^3} + \eta_\infty^2 g \frac{d^2g}{d\xi^2} + \eta_\infty^2 \beta[1 - (\frac{dg}{d\xi})^2], \quad \xi \in [0, 1].$$

Then Figures $1 - 4$ show plot of R_g for g from [VIM]. These figures show the residual of Falkner–Skan equation in domain [0,1]. The MVIM was implemented for Examples $2, 5$ and 9 for the sake of decreasing the maximum values of R_g, as we stated in this paper, this modification is suitable for large values of η_∞ such as $\eta_\infty \geq 4$. Figures $5 - 8$ compare the plot of R_g for VIM and MVIM. These figures show MVIM is more successful than the VIM, because of decreasing the maximum values of R_g.

Now there is a question. Which is more accurate, the new method or the method of [2]? There is an important fact. That is: If we expand solution

Table 3: The values of $R_f(\eta)$ for [VIM] versus [2] in some points of domain for $g(0) = 0$,$\eta_\infty = 2$,$\beta = 0.5$. Note that a(b) means $a \times 10^{-b}$.

$Method$	$\eta_1 = 0.1$	$\eta_2 = 0.4$	$\eta_3 = 0.7$	$\eta_4 = 1$
[VIM]	5.5(11)	1.6103(6)	7.64624(5)	7.935513(4)
[2]	1.36(10)	1.61(6)	7.64487(5)	7.934422(4)

Table 4: The The values of $R_f(\eta)$ for [VIM] versus [2] in some points for $g(0) = 0$,$\eta_\infty = 4$,$\beta = 0.5$. Note that a(b) means $a \times 10^{-b}$.

$Method$	$\eta_1 = 0.1$	$\eta_2 = 0.4$	$\eta_3 = 0.7$	$\eta_4 = 1$
[VIM]	1.92(10)	1.39492(6)	6.680089(5)	7.124317(4)
[2]	1.33(10)	1.46405(6)	6.991498(5)	7.893971(4)

Table 5: The values of $R_f(\eta)$ for [VIM] versus [2] in some points for $g(0) = 0$,$\eta_\infty = 6$,$\beta = 0.5$. Note that a(b) means $a \times 10^{-b}$.

$Method$	$\eta_1 = 0.1$	$\eta_2 = 0.4$	$\eta_3 = 0.7$	$\eta_4 = 1$
[VIM]	8.4(11)	1.57803(6)	7.50259(5)	7.819804(4)
[2]	5.9(11)	1.53769(6)	7.3222(5)	7.671967(4)

$f(\eta)$ of Falkner–Skan in a Taylor series about $\eta = 0$ then all coefficients f_js, in the Taylor series, depend on $f_2 = \frac{f''(0)}{2!} = \frac{\alpha}{2\eta_\infty}$. We can easily calculate as many coefficients f_j as we need. For example:

$$f_0 = \delta, \qquad f_1 = 0, \qquad f_2 = \frac{\alpha}{2\eta_\infty}, \qquad f_3 = \frac{-\beta}{6}, \qquad f_4 = 0, ...$$

Let us truncate the series after 15 terms and assume $f(\eta) = \sum_{j=0}^{15} f_j \eta^j$ is the exact solution of

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta[1 - (\frac{df}{d\eta})^2] = 0, \qquad 0<\eta<\eta_\infty,$$

subject to the boundary conditions

$$f(0) = \delta, \qquad \frac{df}{d\eta}(0) = 0, \qquad \frac{df}{d\eta}(\eta_\infty) = 1.$$

Now we introduce

$$R_f(\eta) = \frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta[1 - (\frac{df}{d\eta})^2].$$

Then we substitute $f(\eta) = \sum_{j=0}^{15} f_j \eta^j$ in R_f and also we substitute the calculated α from the new method and the method of [2] in R_f separately. Tables 3-5 show the values of $R_f(\eta)$ for some η in domain $[0,\eta_\infty]$ for VIM and [2]. In fact these tables show the residual of Falkner–Skan equation for some points in it's domain for VIM and the method of [2] . These results show that almost the accuracy of the new method and the method of [2] are the same. This shows that the results obtained using the method presented in the current paper are in agreement with the results of [2].

5 Conclusion

The well-known He's variational iteration method is a very important tool for solving Falkner–Skan equation. Some modifications are made to find more accurate solutions. In the He's variational iteration method the solution of the problem can be found without discretization [20] of the variables. Therefore, despite of some numerical procedures, instability behavior does not exist. Generally, the present method is very efficient for finding the solution of ordinary differential equations.

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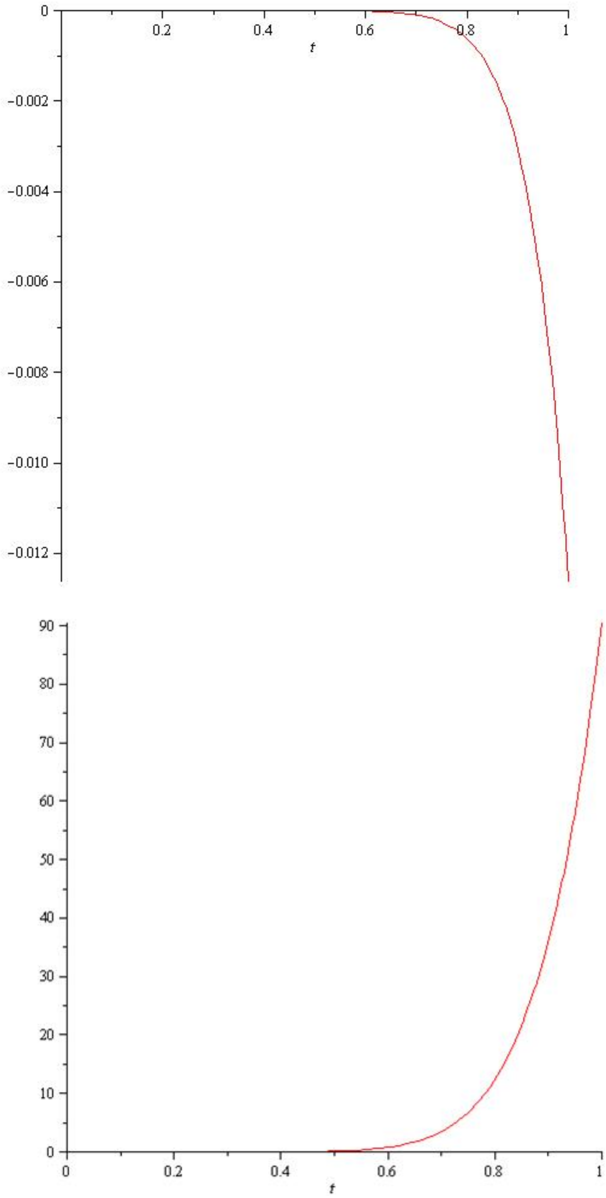


Figure 1: Plot of the $R_g(\xi)$ for g from VIM and $\eta_\infty = 2$, $\beta = 0.5$ and $g(0) = 0$ (up) and plot of the the $R_g(\xi)$ for g from VIM and $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0$ (down) using four terms of method.

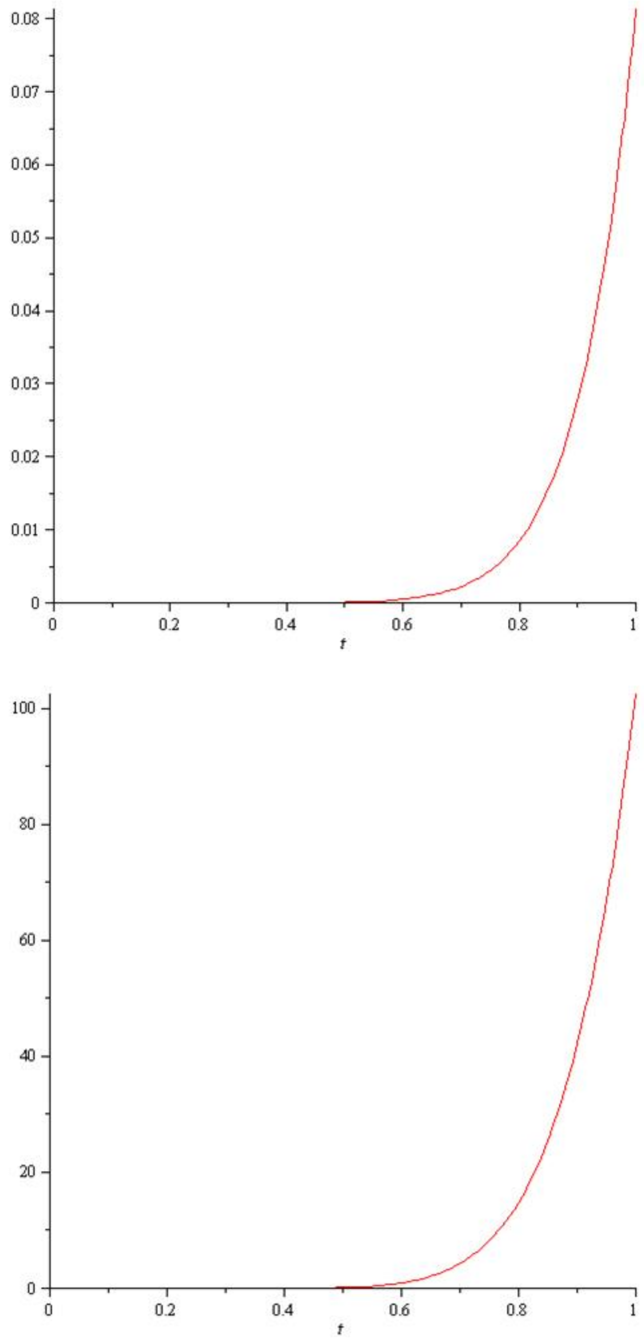


Figure 2: Plot of the the $R_g(\xi)$ for g from VIM and $\eta_\infty = 2$, $\beta = 0.25$ and $g(0) = 0.05$ (up) and plot of the the $R_g(\xi)$ for g from VIM and $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0.02$ (down) using four terms of method.

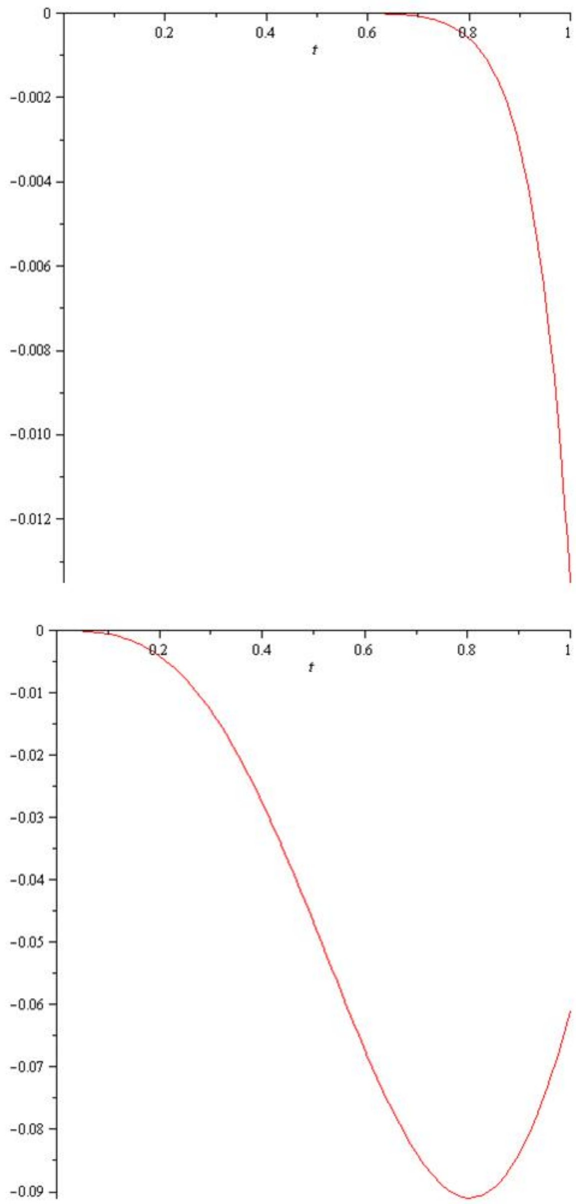


Figure 3: Plot of the the $R_g(\xi)$ for g from VIM and $\eta_\infty = 2$, $\beta = 0.5$ and $g(0) = 0.01$ (up) and plot of the the $R_g(\xi)$ for g from VIM and $\eta_\infty = 2$, $\beta = 0.25$ and $g(0) = -0.5$ (down) using four terms of method.

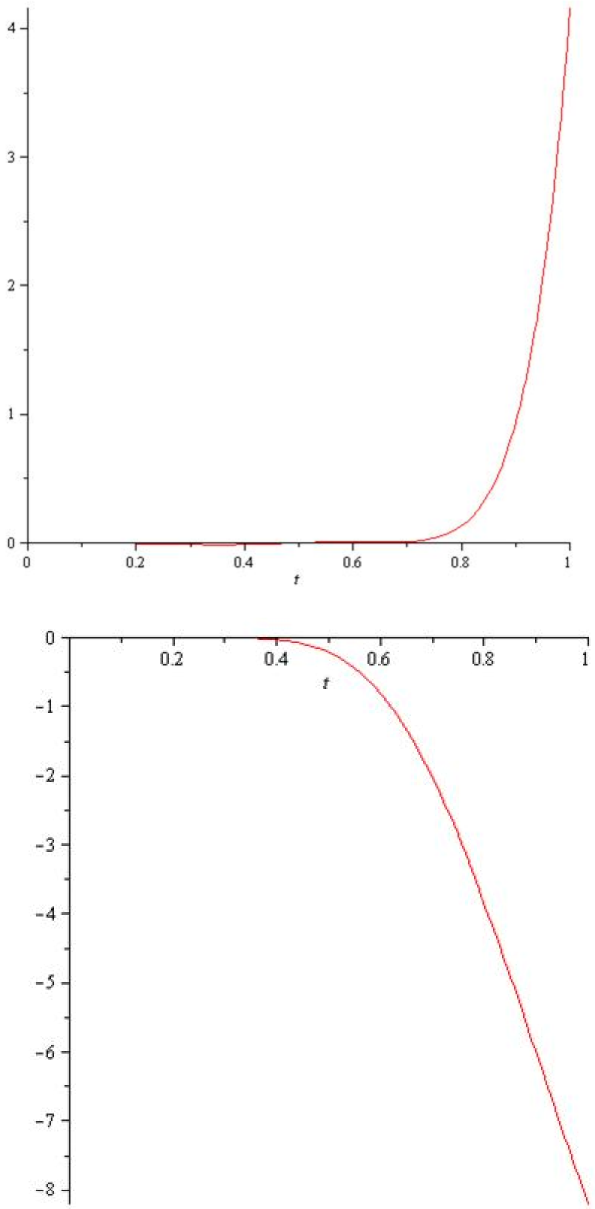


Figure 4: Plot of the the $R_g(\xi)$ for g from VIM and $\eta_\infty = 4$, $\beta = 0.25$ and $g(0) = -0.15$ (up) and Plot of the the $R_g(\xi)$ for $\eta_\infty = 6$, $\beta = 0.5$ and $g(0) = 0$ using [8/8] Padé approximation and four terms of method.

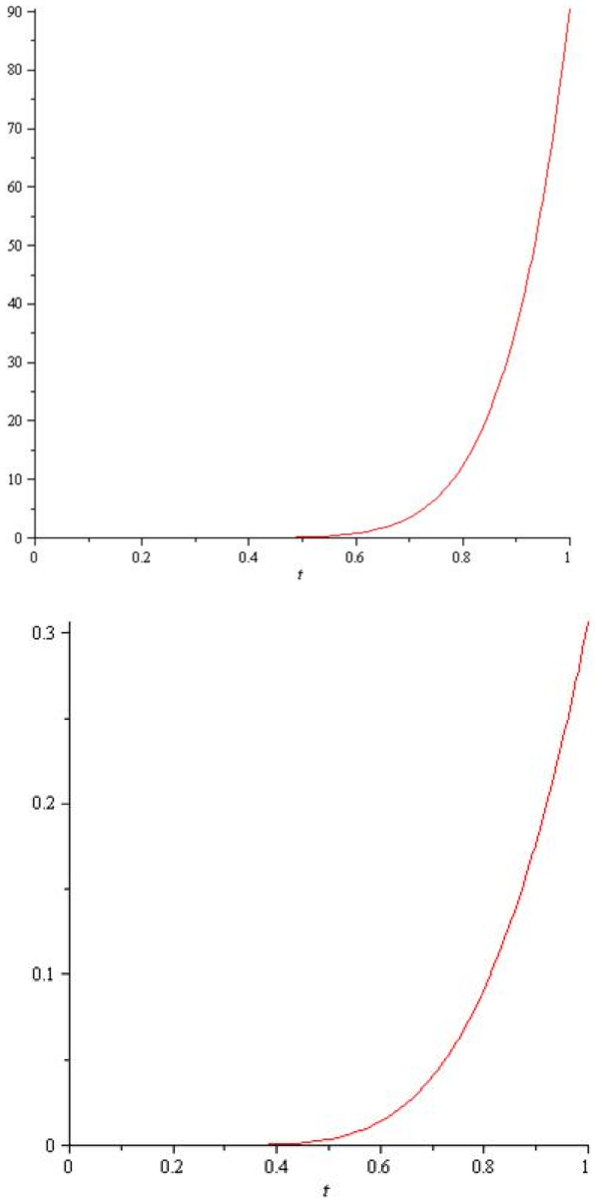
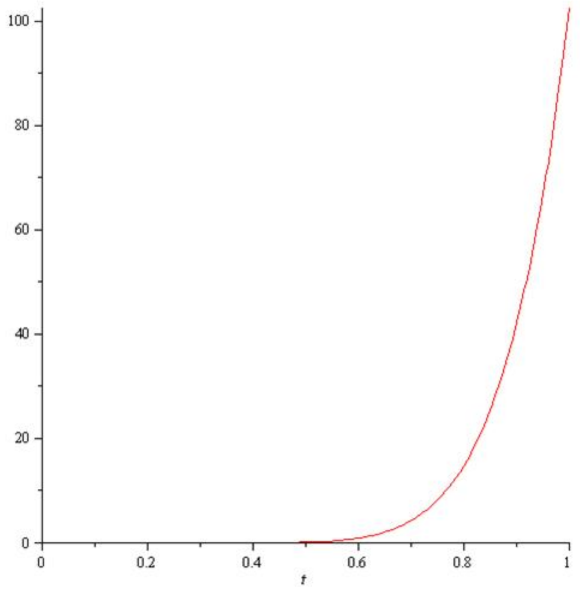


Figure 5: Plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0$ by VIM (up)and Plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0$ by MVIM (down) using four terms of method.



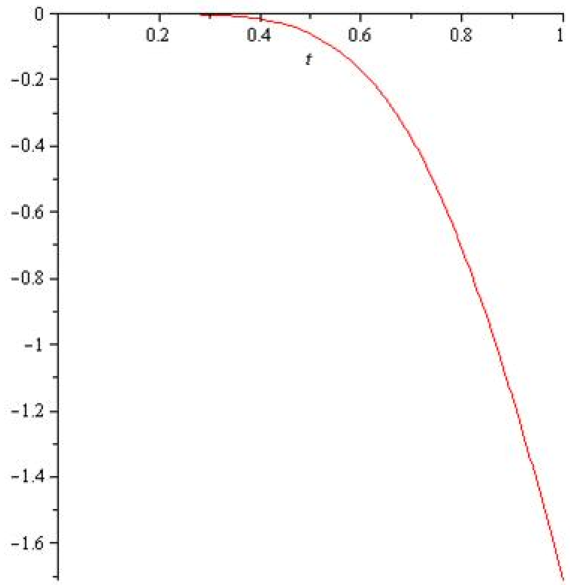
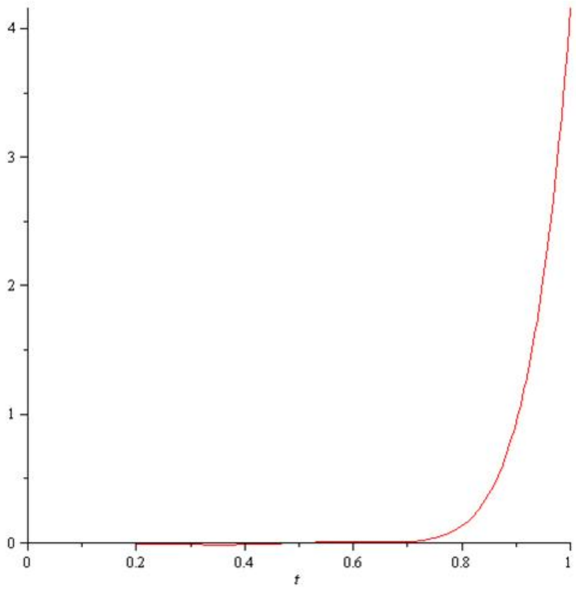


Figure 6: Plot of the $R_g(\xi)$ for $\eta_\infty = 4, \beta = 0.5$ and $g(0) = 0.02$ by VIM (up) and plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0.02$ by MVIM (down) using four terms of method.



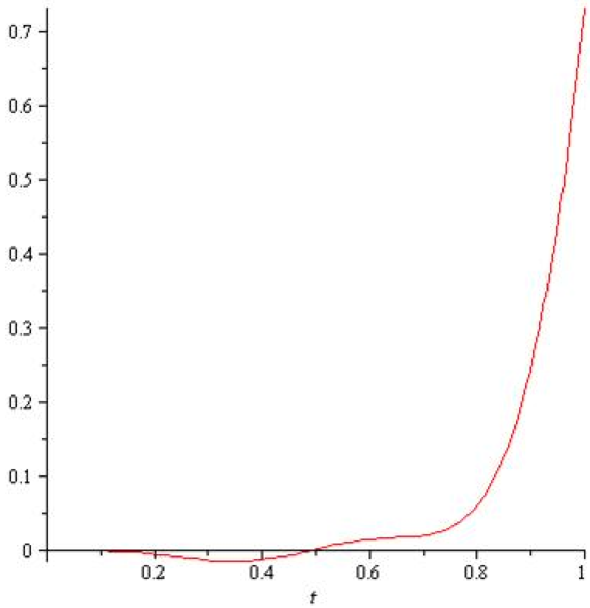


Figure 7: Plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.25$ and $g(0) = -0.15$ by VIM (up) and plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.25$ and $g(0) = -0.15$ by MVIM (down) using four terms of method.

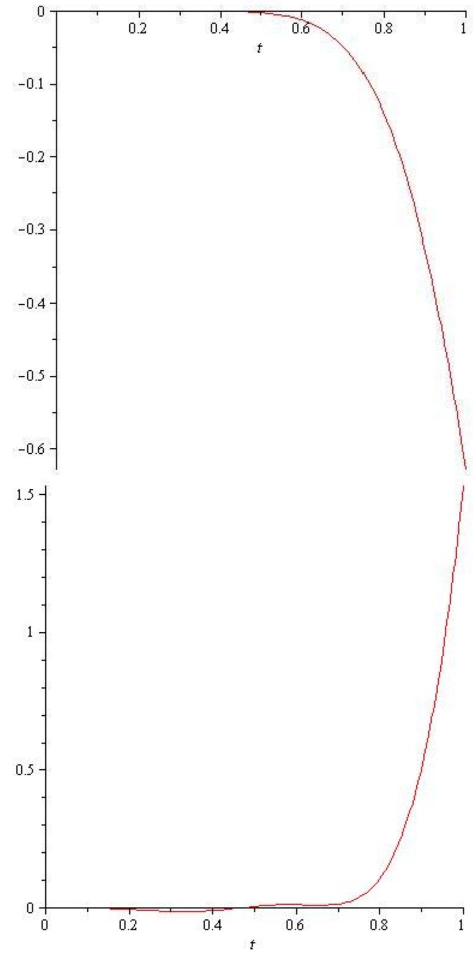


Figure 8: Plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0$ by Padé [8/8] (up) and plot of the $R_g(\xi)$ for $\eta_\infty = 4$, $\beta = 0.5$ and $g(0) = 0.02$ by Padé [8/8] (down) using four terms of method.